

A Second Poincaré Group*

R. Aldrovandi and J. G. Pereira
Instituto de Física Teórica
Universidade Estadual Paulista
Rua Pamplona 145
01405-900 São Paulo
Brazil

Solutions of the sourceless Einstein's equation with weak and strong cosmological constants are discussed by using Inönü-Wigner contractions of the de Sitter groups and spaces. The more usual case corresponds to a weak cosmological-constant limit, in which the de Sitter groups are contracted to the Poincaré group, and the de Sitter spaces are reduced to the Minkowski space. In the strong cosmological-constant limit, however, the de Sitter groups are contracted to another group which has the same abstract Lie algebra of the Poincaré group, and the de Sitter spaces are reduced to a 4-dimensional cone-space of infinite scalar curvature, but vanishing Riemann and Ricci curvature tensors. In such space, the special conformal transformations act transitively, and the equivalence between inertial frames is that of special relativity.

I. INTRODUCTION

The Poincaré group P is naturally associated to Minkowski spacetime M as its group of motions. It contains, in the form of a semi-direct product, the Lorentz group $L = SO(3,1)$ and the translation group T . The latter acts transitively on M and its manifold is just M . Indeed, Minkowski spacetime is a homogeneous space under P , actually the quotient $M = T = P/L$. If we prefer, the manifold of P is a principal bundle $\mathcal{P}(P/L, L)$ with $P/L = M$ as base space and L as the typical fiber [1].

The invariance of M under the transformations of P reflects its uniformity. The Lorentz subgroup provides an isotropy around a given point of M , and the translation invariance enforces this isotropy around any other point. This is the usual meaning of “uniformity”, in which T is responsible for the equivalence of all points of spacetime.

The concept is actually more general. In all local (or tangential) physics, what happens is that the laws of Physics are invariant under transformations related to some specific kind of uniformity of which the above case is but an example (though, quite probably, the most important one). Uniformity includes homogeneity of space and of time, isotropy of space and the equivalence of inertial frames. This holds of course for usual special-relativistic kinematics, but also for Galilean and other non-relativistic kinematics [2], their difference being grounded in their different “kinematical groups”. Most of our experiments are local, and presuppose some such kinematics.

The complete kinematical group, whatever it may be, will always have a subgroup accounting for both the isotropy of space (rotation group) and the equivalence of inertial frames. The remaining transformations will be generically called “translations”, commutative or not. Roughly speaking, the point-set of local spacetime is the point-set of these “translations”. More precisely, kinematical spacetime is defined as the quotient space of the whole kinematical group by the subgroup including rotations and boosts.

Given any solution of Einstein's equation, that is, any acceptable spacetime, it is the local kinematics which will provide the stage-set for local experiments. Our aim in this paper will be to prospect new possible relativistic kinematical groups and spacetimes and reveal an apparently as yet unsuspected case. Our starting point is the well known fact that the Poincaré group can be obtained from the de Sitter group by an appropriate Inönü-Wigner contraction [3]. Such contractions have been first introduced [4] to formalize and generalize the fact that the Galilei group is obtained from the Lorentz group in the non-relativistic limit $c \rightarrow \infty$. The general procedure involves always a preliminary choice of convenient coordinates, in terms of which a certain parameter is made explicit which encapsulates the whole limiting process — the complete new kinematics is obtained by taking that parameter to an appropriate limit. In the specific case of the contraction of the de Sitter to the Poincaré group, the parameter is the

*Dedicated to Abraham Hirsz Zimmerman on the occasion of his 70th birthday.

de Sitter pseudo-radius \mathcal{R} , and the limit is achieved by taking \mathcal{R} to infinity. The curvature of the de Sitter space, which is proportional to \mathcal{R}^{-2} , goes consequently to zero in the limit.

Now, the de Sitter space is a solution of Einstein's equation for an empty space with a cosmological constant $\Lambda = R/4$, where R is the scalar curvature of the de Sitter space. Therefore, the limit of the de Sitter curvature going to zero is equivalent to the limit in which the cosmological constant Λ goes to zero. In this limit, the de Sitter groups reduce to the Poincaré group, and the de Sitter spaces reduce to Minkowski space, a sourceless solution of Einstein's equation with a vanishing cosmological constant. We should mention that, for a small enough cosmological constant, it would be very difficult to differentiate experimentally between a small Λ and a vanishing Λ .

A natural question then arises: what happens in the limit of the de Sitter pseudo-radius \mathcal{R} going to zero, corresponding to a cosmological constant going to infinity? In what follows, we will be concerned mainly with this question. We start by studying the de Sitter groups and spaces. The conformal stereographic coordinates are introduced in such a way to explicitly exhibit the de Sitter pseudo-radius \mathcal{R} . This is necessary to apply the contraction procedure. The contraction limit $\mathcal{R} \rightarrow \infty$ is then studied, which takes the de Sitter group into the Poincaré group, and the de Sitter space into Minkowski space. Proceeding further, the contraction limit $\mathcal{R} \rightarrow 0$ is studied, which is shown to take the de Sitter groups into a kind of “conformal” Poincaré group — the “second Poincaré group” of our title — and the de Sitter spaces into a 4-dimensional cone-space. Finally, in the last section, we discuss a duality relation between the two cases.

II. THE DE SITTER GROUPS AND SPACES

Amongst curved spacetimes, only those of constant curvature can lodge the highest number of Killing vectors. Given the metric signature and the value of the scalar curvature R , these maximally-symmetric spaces are unique [5]. In consequence, the de Sitter spaces are the only uniformly curved 4-dimensional metric spacetimes. There are two kinds of them [6], one with positive, and another one with negative curvature. They can be defined as hypersurfaces in the pseudo-Euclidean spaces $\mathbf{E}^{4,1}$ and $\mathbf{E}^{3,2}$, inclusions whose points in Cartesian coordinates $(\xi^A) = (\xi^0, \xi^1, \xi^2, \xi^3, \xi^4)$ satisfy, respectively,

$$\eta_{AB} \xi^A \xi^B = -(\xi^0)^2 + (\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2 + (\xi^4)^2 = \mathcal{R}^2 ;$$

$$\eta_{AB} \xi^A \xi^B = -(\xi^0)^2 + (\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2 - (\xi^4)^2 = -\mathcal{R}^2 .$$

We use η_{ab} (with indices $a, b = 0, 1, 2, 3$) for the Lorentz metric $\eta = \text{diag}(-1, 1, 1, 1)$ and the notation $\epsilon = \eta_{44}$ to put the conditions together as

$$\eta_{ab} \xi^a \xi^b + \epsilon (\xi^4)^2 = \epsilon \mathcal{R}^2 . \quad (1)$$

The de Sitter space $dS(4,1)$, whose metric can be put into the diagonal form $\eta_{AB} = (-1, +1, +1, +1, +1)$, has the pseudo-orthogonal group $SO(4,1)$ as group of motions. The other, with metric $\eta_{AB} = (-1, +1, +1, +1, -1)$, is frequently called anti-de Sitter space and is denoted $dS(3,2)$ because its group of motions is $SO(3,2)$. The de Sitter spaces are both homogeneous spaces:

$$dS(4,1) = SO(4,1)/SO(3,1) \quad \text{and} \quad dS(3,2) = SO(3,2)/SO(3,1) .$$

The manifold of each de Sitter group is a bundle with the corresponding de Sitter space as base space and $L = SO(3,1)$ as fiber. But the kinematical group is no more a product of groups. If we isolate L and call the remaining transformations “de Sitter translations”, these do not constitute a subgroup and the product of two of them amounts to a Lorentz transformation.

The generators of infinitesimal de Sitter transformations are given by

$$J_{AB} = \eta_{AC} \xi^C \frac{\partial}{\partial \xi^B} - \eta_{BC} \xi^C \frac{\partial}{\partial \xi^A} . \quad (2)$$

They satisfy the commutation relations

$$[J_{AB}, J_{CD}] = \eta_{BC} J_{AD} + \eta_{AD} J_{BC} - \eta_{BD} J_{AC} - \eta_{AC} J_{BD} . \quad (3)$$

The 4-dimensional stereographic coordinates x^μ are given by [7]

$$\xi^a = n(x) \delta^a{}_\mu x^\mu \equiv h^a{}_\mu x^\mu \quad ; \quad \xi^4 = -\mathcal{R} n(x) \left(1 - \epsilon \frac{\sigma^2}{4\mathcal{R}^2} \right) , \quad (4)$$

where

$$n(x) = \frac{1}{1 + \epsilon \frac{\sigma^2}{4\mathcal{R}^2}} , \quad (5)$$

and

$$\sigma^2 = \eta_{\mu\nu} x^\mu x^\nu , \quad (6)$$

with $\eta_{\mu\nu} = \delta^a{}_\mu \delta^b{}_\nu \eta_{ab}$. The $h^a{}_\mu$ introduced in (4) are the components of a tetrad field, actually of the 1-form basis members $\omega^a = h^a{}_\mu dx^\mu = n \delta^a{}_\mu dx^\mu$. The inverse transformations are

$$x^\mu \equiv h_a{}^\mu \xi^a = n^{-1}(\xi) \delta_a{}^\mu \xi^a \quad ; \quad \epsilon \frac{\sigma^2}{4\mathcal{R}^2} = \frac{1 + \xi^4/\mathcal{R}}{1 - \xi^4/\mathcal{R}} , \quad (7)$$

where

$$n(\xi) = \frac{1}{2} \left(1 - \frac{\xi^4}{\mathcal{R}} \right) . \quad (8)$$

In stereographic coordinates, the line element $ds^2 = \eta_{AB} d\xi^A d\xi^B$ is found to be $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$, with

$$g_{\mu\nu} = h^a{}_\mu h^b{}_\nu \eta_{ab} \equiv n^2(x) \eta_{\mu\nu} \quad (9)$$

the corresponding metric tensor. The de Sitter spaces, therefore, are conformally flat, with the conformal factor given by n^2 . We could have written simply $\xi^\mu = n x^\mu$, but we are carefully using the Latin alphabet for the algebra (and flat space) indices, and the Greek alphabet for the homogeneous space fields and cofields. As usual with changes from flat tangent-space to spacetime, letters of the two kinds are interchanged with the help of the tetrad field. This is true for all tensor indices. Connections, which are vectors only in the last (1-form) index, will gain an extra “vacuum” term [8].

The Christoffel symbol corresponding to the metric $g_{\mu\nu}$ is

$$\Gamma^\lambda{}_{\mu\nu} = [\delta^\lambda{}_\mu \delta^\sigma{}_\nu + \delta^\lambda{}_\nu \delta^\sigma{}_\mu - \eta_{\mu\nu} \eta^{\lambda\sigma}] \partial_\sigma (\ln n) . \quad (10)$$

The corresponding Riemann tensor components,

$$R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu{}_{\nu\sigma} - \partial_\sigma \Gamma^\mu{}_{\nu\rho} + \Gamma^\mu{}_{\epsilon\rho} \Gamma^\epsilon{}_{\nu\sigma} - \Gamma^\mu{}_{\epsilon\sigma} \Gamma^\epsilon{}_{\nu\rho} ,$$

are found to be

$$R^\mu{}_{\nu\rho\sigma} = \epsilon \frac{1}{\mathcal{R}^2} [\delta^\mu{}_\rho g_{\nu\sigma} - \delta^\mu{}_\sigma g_{\nu\rho}] . \quad (11)$$

The Ricci tensor and the scalar curvature are, consequently

$$R_{\mu\nu} = \epsilon \frac{3}{\mathcal{R}^2} g_{\mu\nu} \quad (12)$$

and

$$R = \epsilon \frac{12}{\mathcal{R}^2} . \quad (13)$$

In terms of the coordinates $\{x^\mu\}$, the generators (2) of the infinitesimal de Sitter transformations are given by

$$J_{ab} \equiv \delta_a{}^\mu \delta_b{}^\nu (\eta_{\rho\mu} x^\rho P_\nu - \eta_{\rho\nu} x^\rho P_\mu) \quad (14)$$

$$J_{a4} \equiv \epsilon \delta_a{}^\mu \left(\mathcal{R} P_\mu + \frac{\epsilon}{4\mathcal{R}} K_\mu \right) , \quad (15)$$

where

$$P_\mu = \frac{\partial}{\partial x^\mu} \quad \text{and} \quad K_\mu = (2\eta_{\mu\lambda} x^\lambda x^\rho - \sigma^2 \delta_\mu^\rho) P_\rho \quad (16)$$

are respectively the generators of translations and special conformal transformations. For $\epsilon = +1$, we get the generators of the de Sitter group $SO(4, 1)$. For $\epsilon = -1$, we get the generators of the de Sitter group $SO(3, 2)$.

To illustrate the whole process, we will follow the fate of a spinorial test particle in the ensuing contraction procedures by writing the Dirac equation on these spaces. Denoting by γ^d the Dirac matrices, it is given by [9]

$$i \hbar h_d^\mu \gamma^d \left[\partial_\mu - \frac{i}{4} \omega_{ab\mu} \sigma_{ab} \right] \psi(x) - m c \psi(x) = 0, \quad (17)$$

where $h_d^\mu = n^{-1}(x) \delta_d^\mu$ is the inverse tetrad,

$$\omega_{ab\mu} = h^a{}_\rho (\partial_\mu h_b{}^\rho + \Gamma^\rho_{\sigma\mu} h_b{}^\sigma) = (h^a{}_\mu h^b{}_\sigma - h^a{}_\sigma h^b{}_\mu) \partial^\sigma (\ln n) \quad (18)$$

is the spin connection, and

$$\sigma_{ab} = \frac{i}{2} [\gamma_a, \gamma_b] \quad (19)$$

is the spin-1/2 representation of the Lorentz group. A direct calculation shows that

$$n^{-3/2} \gamma^\mu \partial_\mu \left[n^{3/2} \psi(x) \right] + i n M \psi(x) = 0, \quad (20)$$

with $M = mc/\hbar$, and where we have used the notation

$$\gamma^\mu \equiv \delta^\mu_d \gamma^d. \quad (21)$$

III. WEAK COSMOLOGICAL-CONSTANT CONTRACTION

The Inönü–Wigner contraction of the de Sitter to the Poincaré group is obtained by taking the limit $\mathcal{R} \rightarrow \infty$. In this case, it is convenient to rewrite Eqs. (14) and (15) in the form

$$J_{ab} \equiv \delta_a{}^\mu \delta_b{}^\nu L_{\mu\nu} \quad (22)$$

$$J_{a4} \equiv \mathcal{R} \delta_a{}^\mu \Pi_\mu, \quad (23)$$

where

$$L_{\mu\nu} = \eta_{\rho\mu} x^\rho P_\nu - \eta_{\rho\nu} x^\rho P_\mu \quad (24)$$

are the generators of the Lorentz group, and

$$\Pi_\mu = \epsilon \left(P_\mu + \frac{\epsilon}{4\mathcal{R}^2} K_\mu \right) \quad (25)$$

the generators of the de Sitter translations. In terms of these generators, the commutation relation (3) take the form

$$[L_{\mu\nu}, L_{\lambda\rho}] = \eta_{\nu\lambda} L_{\mu\rho} + \eta_{\mu\rho} L_{\nu\lambda} - \eta_{\nu\rho} L_{\mu\lambda} - \eta_{\mu\lambda} L_{\nu\rho}, \quad (26)$$

$$[\Pi_\mu, L_{\lambda\rho}] = \eta_{\mu\lambda} \Pi_\rho - \eta_{\mu\rho} \Pi_\lambda, \quad (27)$$

$$[\Pi_\mu, \Pi_\lambda] = -\epsilon \mathcal{R}^{-2} L_{\mu\lambda}. \quad (28)$$

Proceeding to the contraction limit $\mathcal{R} \rightarrow \infty$, we see that

$$\lim_{\mathcal{R} \rightarrow \infty} L_{\mu\nu} = L_{\mu\nu} \quad \text{and} \quad \lim_{\mathcal{R} \rightarrow \infty} \Pi_\mu = \epsilon P_\mu , \quad (29)$$

and, in consequence, the de Sitter algebra contracts to the usual Poincaré algebra

$$[L_{\mu\nu}, L_{\lambda\rho}] = \eta_{\nu\lambda} L_{\mu\rho} + \eta_{\mu\rho} L_{\nu\lambda} - \eta_{\nu\rho} L_{\mu\lambda} - \eta_{\mu\lambda} L_{\nu\rho} , \quad (30)$$

$$[P_\mu, L_{\lambda\rho}] = \eta_{\mu\lambda} P_\rho - \eta_{\mu\rho} P_\lambda , \quad (31)$$

$$[P_\mu, P_\lambda] = 0 . \quad (32)$$

We see also that

$$\lim_{\mathcal{R} \rightarrow \infty} g_{\mu\nu} = \eta_{\mu\nu} ,$$

which shows that this limit leads exactly to the Minkowski geometry, a geometry gravitationally related to a zero cosmological constant. Correspondingly, the Dirac equation (20) acquires the expected form,

$$\gamma^\mu \partial_\mu \psi(x) + i M \psi(x) = 0. \quad (33)$$

IV. STRONG COSMOLOGICAL-CONSTANT CONTRACTION

Let us consider now the opposite limit, that is, $\mathcal{R} \rightarrow 0$. In this case, we rewrite Eqs. (14) and (15) in the form

$$J_{ab} \equiv \delta_a^\mu \delta_b^\nu L_{\mu\nu} \quad (34)$$

$$J_{a4} \equiv \mathcal{R}^{-1} \delta_a^\mu \kappa_\mu , \quad (35)$$

where $L_{\mu\nu}$ are the generators of the Lorentz group, and

$$\kappa_\mu = \frac{1}{4} K_\mu + \epsilon \mathcal{R}^2 P_\mu . \quad (36)$$

In terms of these generators, the commutation relation (3) becomes

$$[L_{\mu\nu}, L_{\lambda\rho}] = \eta_{\nu\lambda} L_{\mu\rho} + \eta_{\mu\rho} L_{\nu\lambda} - \eta_{\nu\rho} L_{\mu\lambda} - \eta_{\mu\lambda} L_{\nu\rho} , \quad (37)$$

$$[\kappa_\mu, L_{\lambda\rho}] = \eta_{\mu\lambda} \kappa_\rho - \eta_{\mu\rho} \kappa_\lambda , \quad (38)$$

$$[\kappa_\mu, \kappa_\lambda] = -\epsilon \mathcal{R}^2 L_{\mu\lambda} . \quad (39)$$

In the contraction limit $\mathcal{R} \rightarrow 0$, one can see that

$$\lim_{\mathcal{R} \rightarrow 0} L_{\mu\nu} = L_{\mu\nu} \quad ; \quad \lim_{\mathcal{R} \rightarrow 0} \kappa_\mu = \frac{1}{4} K_\mu , \quad (40)$$

and consequently the de Sitter algebra contracts to

$$[L_{\mu\nu}, L_{\lambda\rho}] = \eta_{\nu\lambda} L_{\mu\rho} + \eta_{\mu\rho} L_{\nu\lambda} - \eta_{\nu\rho} L_{\mu\lambda} - \eta_{\mu\lambda} L_{\nu\rho} , \quad (41)$$

$$[K_\mu, L_{\lambda\rho}] = \eta_{\mu\lambda} K_\rho - \eta_{\mu\rho} K_\lambda , \quad (42)$$

$$[K_\mu, K_\lambda] = 0 . \quad (43)$$

The Lie group corresponding to this algebra, denoted by Q and formed by a semi-direct product of Lorentz and special conformal transformations, is completely different from P but presents the same Lie algebra as the Poincaré

group. It will rule the local kinematics of high Λ spaces. We see also that, for $\mathcal{R} \rightarrow 0$, the asymptotic behaviour of the conformal factor $n(x)$ is

$$n(x) \sim \epsilon \frac{4\mathcal{R}^2}{\sigma^2} .$$

The metric tensor, thus, becomes singular,

$$\lim_{\mathcal{R} \rightarrow 0} g_{\mu\nu} = 0 ,$$

and the Riemann and Ricci curvature tensors vanish, as can be seen from Eqs. (11) and (12), respectively. However, the scalar curvature R becomes infinity, which is in accordance with the fact that in this limit the cosmological constant goes to infinity. We can conclude, therefore, that the contraction limit $\mathcal{R} \rightarrow 0$ leads both de Sitter spaces to a spacetime, denoted by N , whose geometry is gravitationally related to an infinite cosmological constant. It is a 4-dimensional cone-space in which $ds = 0$, and whose group of motion is Q . Analogously to the Minkowski case, N is also a homogeneous space, but now under the kinematical group Q , that is, $N = Q/L$. In other words, the point-set of N is the point-set of the special conformal transformations. Furthermore, the manifold of Q is a principal bundle $\mathcal{P}(Q/L, L)$, with $Q/L \equiv N$ as base space and L as the typical fiber.

The kinematical group Q , like the Poincaré group, has the Lorentz group L as the subgroup accounting for both the isotropy and the equivalence of inertial frames in this space. However, the special conformal transformations introduce a new kind of homogeneity. Instead of ordinary translations, all the points of N are equivalent through special conformal transformations.

This 4-dimensional cone-space should not, of course, be confused with the 3-dimensional light-cone of special relativity. Nevertheless, due to the conformal factor $n(x)$, the mass term in (20) vanishes and the Dirac equation takes the Weyl form

$$\sigma^3 \gamma^\mu \partial_\mu [\sigma^{-3} \psi(x)] = 0 , \quad (44)$$

which can be reduced to a two-component equation. It is not surprising that, in the presence of the newly-acquired conformal symmetry, the mass come to pass out of sight.

V. FINAL REMARKS

By the process of Inönü–Wigner group contraction with $\mathcal{R} \rightarrow \infty$, both de Sitter groups are reduced to the Poincaré group P , and both de Sitter spacetimes are reduced to the Minkowski space M . As the de Sitter scalar curvature goes to zero in this limit, we can say that M is a spacetime gravitationally related to a vanishing cosmological constant. On the other hand, in a similar fashion but taking the limit $\mathcal{R} \rightarrow 0$, both de Sitter groups are contracted to the group Q , formed by a semi-direct product between Lorentz and special conformal transformation groups, and both de Sitter spaces are reduced to the cone-space N , which is a space with vanishing Riemann and Ricci curvature tensors. As the scalar curvature of the de Sitter space goes to infinity in this limit, we can say that N is a spacetime gravitationally related to an infinite cosmological constant. If the fundamental spacetime symmetry of the laws of Physics is that given by the de Sitter instead of the Poincaré group, the P -symmetry of the weak cosmological-constant limit and the Q -symmetry of the strong cosmological-constant limit can be considered as limiting cases of the fundamental symmetry.

Minkowski and the cone-space can be considered as *dual* to each other, in the sense that their geometries are determined respectively by a vanishing and an infinite cosmological constants. The same can be said of their kinematical group of motions: P is associated to a vanishing cosmological constant and Q to an infinite cosmological constant. The *dual* transformation connecting these two geometries is the spacetime inversion

$$x^\mu \rightarrow -\frac{x^\mu}{\sigma^2} .$$

Under such a transformation, the Poincaré group P is transformed into the group Q , and the Minkowski space M becomes the cone-space N . The points at infinity of M are concentrated in the vertex of the cone-space N , and those on the light-cone of M becomes the infinity of N . It is interesting to notice that, despite presenting an infinite scalar curvature, the concepts of space isotropy and equivalence between inertial frames in the cone-space N are those of special relativity. The difference lies in the concept of uniformity as it is the special conformal transformations, and not ordinary translations, which act transitively on N .

Besides presenting an intrinsic mathematical and physical interest, in the light of the recent supernovae results [10] favoring possibly quite large values for the cosmological constant, the above results may acquire a further relevance to Cosmology, with applications to inflationary models as well.

ACKNOWLEDGMENTS

The authors would like to thank FAPESP–Brazil and CNPq–Brazil for partial financial support.

- [1] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry* (Interscience, New York, 1963).
- [2] H. Bacry and J.-M. Lévy-Leblond, J. Math. Phys. **9**, 1605 (1967).
- [3] E. İnönü and E. P. Wigner, Proc. Natl. Acad. Scien. **39**, 510 (1953). See also: R. Gilmore, *Lie Groups, Lie Algebras, and Some of Their Applications* (J. Wiley, New York, 1974).
- [4] E. İnönü, in F. Gürsey (ed.), *Group Theoretical Concepts and Methods in Elementary Particle Physics*, Istanbul Summer School of Theoretical Physics (Gordon and Breach, New York, 1962).
- [5] S. Weinberg, *Gravitation and Cosmology* (J. Wiley, New York, 1972).
- [6] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space–Time* (Cambridge University Press, Cambridge, 1973).
- [7] F. Gürsey, in F. Gürsey (ed.), *Group Theoretical Concepts and Methods in Elementary Particle Physics*, Istanbul Summer School of Theoretical Physics (Gordon and Breach, New York, 1962).
- [8] See, for example, R. Aldrovandi and J. G. Pereira, *An Introduction to Geometrical Physics* (World Scientific, Singapore, 1995).
- [9] P. A. M. Dirac, in *Planck Festschrift*, edited by W. Frank (Deutscher Verlag der Wissenschaften, Berlin, 1958).
- [10] S. Perlmutter *et al*, Nature **391**, 51 (1998); A. G. Riess *et al*, Ap. J. (1998), in press (astro-ph/9805201).